Math Review

Summer 2016

*Topic 2*

1. Basic topology of the reals

2.1. Sets, sequences and limits

Sets:

As you may recall, the basic set notations are:

: Union

: Intersection

: Complement

Everything not in A

: Difference

×: Cartesian product

If some are feeling rusty, the Cartesian product of two sets is the set of all possible ordered pairs whose first component is a member of the first set and whose second component is a member of the second set.

Here, I quickly touch on the definitions for supremum and infimum.

Supremum and Infimum:

First, an **ordered set**, , is a set in which in an order is defined. An **order** on, denoted <, has the following properties:

(i) If and then one, and only one, of the following statements is true:

(ii) For , if and , then

Let be an ordered set and

**Upper bound**: If there exists a such that for every then we say that is **bounded above**, and call an **upper bound** of .

**Lower bound**: Now, if there there exists an such that for every , then we say that is **bounded below** and call the **lower bound**.

*Example*. Let .

Then −2 is a lower bound for both and

Any number you pick in the sets and you will see that

What would be an upper bound for both and ?

For instance, 2.

Anything different about bounds for sets As we note, the subset is not bounded below and the subset is not bounded above.

Let be an ordered set and

* Suppose is an upper bound of If for all there exists an such that then is called the **least** **upper bound** of or the supremum of The supremum can be expressed as:

This can be viewed as the smallest number among those upper bounds (*l.u.b*)

Now, not looking at specific given but more the set of that fits our definition. The upper bound foris .

What is ? We are looking at all numbers in (. Do we have a number such that such that Yes, 1!

* Suppose is the lower bound of . If for all there exists an such that , then we say that is the **greatest lower** **bounded** of or the infimum of The infimum can be expressed as:

This can be viewed as the largest number among those lower bounds (*g.l.b*)

Now, not looking at specific given but more the set of that fits our definition. The lower bound foris .

What is ? We are looking at all numbers in ( . Do we have a number such that such that Yes, -1!

We call these

For any given subset can have at most one and one . If is not bounded above then we say that

Sequences and limits:

A sequence is a function, defined on the set of natural numbers, . We have for and usually denote the entire sequence by the symbol or . The values of or , are called the terms of the sequence.

Can we think of a couple of example?

(i) (iii)

(ii) (iv)

Definition: A sequence, **converges to a** , , if given , there is some (i.e. element in the sequence), such that whenever

We can also say that

*Example*. An example can help with the definition. Consider the sequence:

Is this converging or diverging?

Clearly it is getting closer and closer to 0, which is the limit for this sequence. The distance between that and the limit L is less than the small number

A sequence, **diverges** if it does not converge.

Theorem: If the sequence converges then the limit of is unique.

Consider the following properties of sequences:

Suppose that for the real number sequences and we have and . Then:

(i)

(ii)

(iii)

(iv) if for all we have .

Cartesian Plane:

The set of real numbers is denoted by the symbol and is defined as:

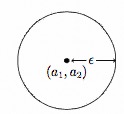
Considering the set product . Any point in the set (any pair of numbers) can be identified with a point in the Cartesian plane.

2.2 Open and Closed Sets

You will use the term neighborhood or a ball around a point often in Micro. This is really getting at limits. A limit is special case of an accumulation point (or cluster point): A point is an accumulation point of a sequence, , if for every ball around , there are infinitely many elements of the sequence, , where .

What is a ball around a point?

Building on the last point, a ball around point of radius is all points such that:

This ball is the local neighborhood of a point . This can be illustrated better with an figure in and .

More generally, any n-tuple, or vector, is just an n-dimensional ordered tuple and can be thought of as a "point" in n-dimensional Eucledian space or "n-space". A n-space is defined as the set product:

How do we define ? It may be useful to define the subset denoted by

Metric space and sets:

A metric space is a set with a notion of distance defined among the points within the set. For any two points and in , denote the distance between them as

.

You may recall that in , this takes the form of:

In , this takes the form of:

Pulling everything together, we are ready to tackle some key definitions:

Open The open - ball with center and radius (a real number) is the subset of points in :

Closed The closed - ball with center and radius (a real number) is the subset of points in : .

Open sets in : is an open set if, for all , there exists some such that

Open sets in is a closed set if its complement, is an open set

2.3. Convex Set

What is a convex set?

A set is convex iff we can connect any two points in the set by a straight line that lies entirely within the set. is a convex set if for all and , we have:

we have , for all t in the interval

We go through a simple example here and we will do a slightly more applicable example from the *Production mini* at the end of this class.

*Example.* Let . Consider any two points in the set, and Define the convex combination . For any value of for any two points in , we have . If .

*Example.* Consider two vectors in denoted by and .

How would you define this convex combination?

For any value of for any two points in , we have .

*Example*. The Intersection of Convex Sets is Convex. Let and be convex

sets in . Then is a convex set.

Proof. Let and be convex sets. Let and be any two points in .

Because and . Because and .

Let , for be any convex combination of and .

Because S is a convex set, .

Because is a convex set, .

Because and , .

**Recall that z, and are all arbitrary, they could be any point!**

Now, because every convex combination of any two points in is also in , is a convex set.

2.4 Bounded sets in

A set in is called bounded if it is entirely contained within some (either open or closed). That is, is bounded if there exists some such that for some .

2.5 Compact sets

One can think of the concept of boundedness basically means that the set has finite size.

Then, a set in is called compact if and only if it is closed and bounded.

There are many definitions for compact set, this one is sufficient for us. You may have to learn some other definitions if you take Macro.

2.6 Functions and properties of relations

Binary relation: A binary relation is any collection of ordered pairs between the sets and .

Let be the colors {red, yellow, green}, and the set of fruits {apple, banana, pear}. The statement “is the color or” defines the relation between the two sets. We denote this as .

The following definitions are critical for your understanding of many concepts in Micro theory.

*Completeness.* A relation on is complete if, for all elements and in , or .

*Transitivity*. A relation on is transitive if, for any three elements and in and , implies .

In consumer theory, you will come across something called a preference relation, which is binary in nature. For example, if you say , you mean is preferred to

Quick : What is equivalent to here from the above definition?

Professor Glewwe will cover this part in details and my example is quite simple.

: You have the following set . Your preference relationships are given by: , , and .

Is it complete? How about transitive?

: For any given element and in , we have or . So yes, completeness is satisfied.

We have and which imply that . However, we find that , so transitivity is not met.

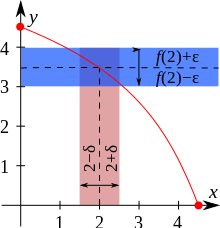
Function: A function is a relation that associates each element of one set with a single, unique element of another set. We say that the function is a mapping from one set to another set and write .

2.7 Continuity

What isa continuous function?

Continuous function: A function is continuous at a point at if and only if, for

every , there exists , such that:



Intuitively, a function is continuous if you can trace the graph of the entire function without ever lifting your pencil from the page. For your purposes, apply the following theorem to show continuity:

Theorem: The function is continuous at if and only, for every sequence , we have . The function is then continuous on if and only if

it is continuous for all This definition is clearer with an exercise/example:

Show that that the function is continuous using the above theorem

Hint: Start by taking the limit of

:

What are some other properties of continuous functions?

Other properties of continuous functions are as follows:

Let and be continuous functions in. Then:

(i)

(ii) where

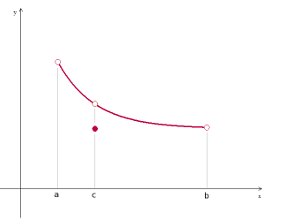
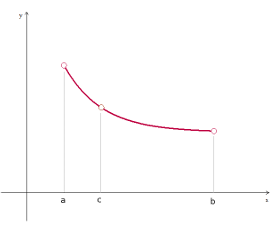
(iii)

(iv) (composition of two functions)

are also *continuous.*

: Can we think of examples where a function, say is *not continuous* a point *?*

*: is not defined ; does not exist ;*



We now have all the tools to be able to try a more relevant example of ***convex set*** from Production. There are many more examples that you now equipped to try, but time is of essence here. Plus, you will the next year to see many of these yourself. The rationale of this part is show you that what we are currently reviewing translate directly into material that you see and learn in the coming year.

Couple of definitions that we will discuss to make sure we are catching notations correctly:

Vector of outputs

Vetor of inputs/factors

**PPS** is the Production Possibilities Set, that is the technology.

**IRS** is the Input Requirement Set or all combinations of input capable of producing a combination of output. It is defined as follows:

You are told that the input requirement set is *convex*.

Write out a one sentence statement to demonstrate that. No need for a proof. I would encourage you to try it out before reading through the hints. Use one hint at a time and see what you come up with!

Hint 1: If you have not already, read section 2.3 again.

Hint 2: What is in this statement?

Hint 3: Can you introduce a couple of new variables that can help you with your statement?

Hint 4: How about and

Hint 5: Have you defined a here yet?

Hint 5: How would you define now that you have a? Can you put it all together?

Hint 6: Now you have the answer:

For all , and all ,

Exercise for home:

Let a collection of commodities vectors be given as:

You can view commodities as food, clothing, entertainment etc. Each will have multiple elements, for instance food may contain dairy, candy, fresh fruits, and so on. We can call an element of this set as a “bundle.”

1. One feature of this consumption set is that it is convex. Define this convex set.

Denote and as two elements in the set. Then let such that is also an element of the set.

2. Let price be and wealth be . Then a person’s budget is given by .

Let the budget set be defined as: . Show that this budget set is convex. (*Hint: use the fact that you know that the consumption set is convex to begin with*).

Let and be two elements in the set. Since the consumption set is convex, then evidently . What do you want to show? , ie:

Going back to our definition:

and . Further,

and (just multiplying with the relevant scalars)

Putting these two together:

Since , ths budget set is convex!